Arbitrary and Necessary Part 1: a Way of Viewing the Mathematics Curriculum

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I start with a proposition:

> *If I'm having to remember ..., then I'm not working on mathematics.*

and follow it with an anecdote. Decide how you would answer Katie's question before reading on.

> Katie was about four years old when her mother, Barbara, mentioned New York while talking to someone else.
>
> Katie: Where is New York?
> Barbara: In the United States.
> Katie: Why?

When I heard this interchange, I was struck by the simplicity of the question and the difficulty I felt in deciding how to provide an 'answer'. Barbara did give a response, which in some ways reflected what I felt - "Because it just is". An alternative to Barbara's response could be to say it was when people were in the United States that they decided to name their town *New* York after the English town *York*, which some of them had known. New York, as far as Katie was concerned, could be anywhere. There could be no reason for her to know that it *had* to be in the United States, because it does not *have* to be: it just so happens that it is. Katie could only find out where New York is by being informed by someone who already knows, or by gathering the information from some other source - be it a book, television, map or whatever.

It is similar for me if I am asked someone's name, someone whom I have never met or heard of before. If they are within my sight, I can look at them and wonder what their name is, seeing whether I can guess it. If I do guess, I have to wait to see whether it is confirmed or not. To know someone's name, I have to be told, and even then I am in a position of having to trust that the person telling me is not lying. Even when I do have that trust, learning a person's name will require work from me to remember it and to associate it with that particular person. Such things are in the realm of memory. If I am going to know someone's name, then I will need to be informed of what it is, and I will need to memorise it if I am going to be in a position to know it again at a later time.

I can make up my own name for this person: however, I will have difficulty in referring to this person when communicating with other people, since we do not share the same referent. So making up my own name may have some interest for me, but it will be of little use when communicating with others. So, I am back to needing to be informed of this person's name and memorising it for later use.

What a mathematical equivalent might be of the Katie anecdote: again, consider how you would respond before reading on.

> Student: How many sides has a square got?
> Teacher: Four.
> Student: Why?

The only reason why a square has four sides is that a decision was made a long time ago to call four-sided shapes with particular properties 'squares'. There is nothing about these shapes which means that they *have* to be called squares - indeed, in other languages, the same shapes are given different names. Looking at the shapes carefully is not going to help a student to know what the name of the shapes is, just as looking at the person does not reveal what their name is. All names, within mathematics or elsewhere, are things which students need to be informed about, and part of a teacher's role is to inform students of such things.

Once students are informed, there is more work the students still need to do. They have to memorise the word and associate the word with shapes with those particular properties. It is typical of the realm of memory that not only has a word, for example, to be memorised, but that word also has to be associated with the right things. Many times students successfully remember a word but may not have made the appropriate association. For example, a student may not call Figure 1 a square, since the properties they associate with *square* do not include sides which are not horizontal or vertical.

Even names which are systematic and rule-based, such as 'octagon', 'heptagon' and 'hexagon', which are generated from particular linguistic roots, do not lead a student to *know for sure* that a five-sided polygon will be called 'pentagon', as the example of 'square' shows, since a four-sided polygon

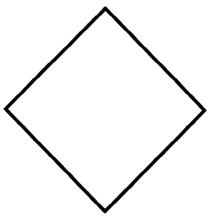

Figure 1 A square?

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is not called a 'tetragon'. Since names are socially and culturally agreed, then someone within that culture will have to inform a novice as to whether their - quite sensible - guesses of 'pentagon' and 'tetragon' are indeed the names accepted within that culture, and so deemed to be 'correct'.

Arbitrary

Names and labels can feel arbitrary for students, in the sense that there does not appear to be any reason why something *has* to be called that particular name. Indeed, there is no reason why something has to be given a particular name. Ginsburg (1977) gives a transcript of a conversation with a second grader, Kathy:

> I: Why do you write a 13 like that, a 1 followed by a 3?
>
> K: 'Cause there's one 10, right? So you just put 1. I don't know why it's made like that. They could put 10 ones and a 3. So you see 13 is like 10 and 3, but the way we write it, it would be 103 so they just put 1 for one 10 and 3 for the extra 3 that it adds on to the 10. (p. 88)

Kathy shows an awareness that the symbolic way in which numbers are written is a choice and does not have to be the way that it is. She even offers an alternative. Names (I include labels and symbols under this heading for convenience of writing) are about choices which have been accepted within a particular community. If a student wishes to become part of the same community, then the student needs to accept that name, rather than *question* it.

I describe something as *arbitrary* if someone could only come to know it to be true by being informed of it by some external means - whether by a teacher, a book, the internet, etc. If something is arbitrary, then it is arbitrary for all learners, and needs to be memorised to be known. Gattegno (1987) claimed:

> there is knowledge that is distinguished sharply from awareness - the knowledge solely entrusted to one's memory, such as the label for such an object or a telephone number, items which are arbitrary. Without someone else, that knowledge would not exist for us. (p. 55)

It is not only labels, symbols or names which are arbitrary. The mathematics curriculum is full of conventions, which are based on choices which have been made at some time in the past. For anyone learning those conventions today, they may seem arbitrary decisions. For example, why is the *x* co-ordinate written first and the *y* co-ordinate second? This is only a convention, and there is no reason why *x* must be first. As a consequence, a student might say that they will write the *y* first then! The issue of students wanting to do things their way and not accepting a cultural convention can prove a point of tension for a teacher, since there is no reason a teacher can offer for why this convention *must* be so. Some phrases are offered, such as "walk along the hall and then up the stairs" or "x comes before y in the alphabet". However, these are stories invented by teachers mainly as memory aids rather than justifications. The truth is that there is no reason why *x* must come first.

I recall a time I was playing snooker with my five-year-old nephew, Robert, on a small snooker table and had just potted the yellow. I was next to the green and was snookered on the brown. Robert said that the brown was next. I said that the green was next. He insisted on the brown. There was no reason I could offer to justify the green being next; there was nothing in the colours which meant that green must come after yellow, and this was a convention that Robert was not going to accept, particularly since it was to his advantage not to do so! Getting students to accept and adopt names and conventions is not always easy.

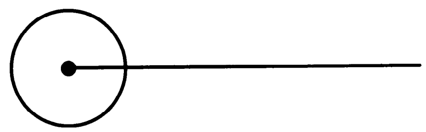


Figure 2 Teacher: "As can be observed from this diagram, a whole turn must be divided up into 360".

The fact that 360 was chosen as the number of units in a whole turn had as much to do with the people who were making such a decision, and what they were aware of at the time, as for any other reason. The Babylonians had a numeration system based on 60 and were looking at the ratio of the perimeter of a regular hexagon to the circumference of a circle. Knowing that the perimeter of a regular hexagon is six times the radius of the circumscribed circle apparently led to the circle being divided into 6 x 60, that is 360, degrees. If such a decision were to be made today, with our metric system of measurements, perhaps 100 would have seemed just as natural. For a pupil in a classroom, who lives in today's world, 360 is by no means an obvious choice. A student cannot look closely at a whole turn, analyse it and come to the conclusion that a whole turn must be split up into 360 units (see Figure 2).

As Clausen (1991) comments:

> I have long felt that our use of degrees to measure amounts of turn (angles) is very arbitrary. There is no way that a child (or adult) can *intuit* that there are 360 degrees in a whole turn. This is totally arbitrary, formal, true-because-Teacher-says-so knowledge. (p. 16, emphasis in original)

A teacher has to inform a learner about how many degrees there are in a whole turn. One attempt to make it feel less arbitrary is to bring a historical perspective into the classroom. Students may learn some history from this approach, but they will not be developing mathematically through memorising the arbitrary. As Pimm (1995) observes:

> The way of thinking embodied in the simple phrase 'it is or it is not' is profoundly mathematical. (p. 187)

However, *it could be* is not so profoundly mathematical!

And the arbitrary is full of *could be*. In contrast to the arbitrary nature of degrees, Clausen goes on to observe:

> On the other hand, the idea of half-turns, quarter-turns, two-thirds turns, and so on, has a real, valid meaning in itself. If I turn right round, so that I am facing the way I was to start with, then I know that I have done a whole turn. There is nothing new to learn – no arbitrary number, like 37, for Teacher to tell me. And I can work out for myself what half-turns, quarter-turns and so on mean, using my own experience of turning half-way round, or a quarter round, or whatever. Fractions of a turn have an intuitive validity which the concept of degrees lacks. (p. 16)

Here Clausen highlights the 'it is or it is not' nature of fractions of turns which can be known without being informed by a teacher.

Necessary

There are aspects of the mathematics curriculum where students do not need to be informed. These are things which students can work out for themselves and know to be correct. They are parts of the mathematics curriculum which are not social conventions but rather are properties which can be worked out from what someone already knows. As Clausen pointed out, there are properties about the act of turning which I can know for myself. For example, if I turn a quarter turn and then a quarter turn again, I have made a half turn. It is possible to find out about other fractions of a whole turn without having to be informed. So, the mathematical content which is on a curriculum can be divided up into those things which are arbitrary and those things which are necessary.

All students will need to be informed of the arbitrary. However, the necessary is dependent upon the awareness students already have: for example, it is necessary that the length of the required side of the triangle in Figure 3 is $\frac{\sqrt{3}}{2}$. However, not every student will be in a position to be aware of this. So although this is necessary, it does not imply that all students have the awareness to be able to work this out, only that *someone* is able to work this out without the need to be informed of it.

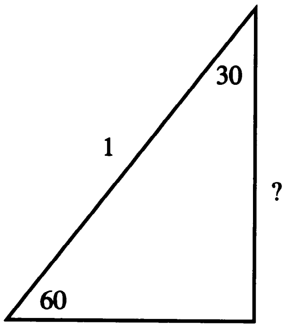


Figure 3 What is the length of the side?

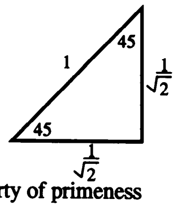
Those things which are necessary can be worked out: it is only a matter of whether particular students have the awareness required to do so. If not, then maybe it is not the best choice of topic to be taught at that moment in time. For example, I did not choose to teach integration of trigonometric functions to the majority of eleven-year-olds I taught. If a student does have the required awareness for something, then I suggest the teacher's role is not to inform the student but to introduce tasks which help students to use their awareness in coming to know what is necessary. What is necessary is in the realm of awareness, whereas the arbitrary is in the realm of memory (see Figure 4).

Arbitrary	*All* students *need* to be informed of the arbitrary by someone else	*Realm of memory*
Necessary	*Some* students *can* become aware of what is necessary without being informed of it by someone else	*Realm of awareness*

Figure 4 Arbitrary and necessary

Viewing the curriculum

I asked my student teachers to write down a list of those things in the mathematics curriculum which cannot be worked out (might be so), and those things which can be worked out (must be so). They came up with the list given in Figure 5.

<table>
<tr><th>Cannot be worked out
(might be so)</th><th>Can be worked out
(must be so)</th></tr>
<tr><td>Names of shapes
Definitions of ...
Measuring bearings from north
x and y co-ordinates
How heavy is a kg?
How long is a metre?
Terminology – e.g. names of theorems, such as 'factor' theorem
word/label</td><td>Interior angles of regular polygons
$V = IR \Rightarrow I = \frac{V}{R}$
Solution of a linear equation
What happens to numbers if multiplied by < 1 or > 1
Rough estimates of measurements
2 x 3
Finding factors of $a^3 + b^3$
Finding angles or lengths in triangle problems, for example based upon this triangle:


Property of primeness
Symmetry</td></tr>
<tr><td>Summarised as: words, symbols, notation and conventions.</td><td>Summarised as: properties and relationships.</td></tr>
</table>

Figure 5 A way to divide the curriculum

Such division of the mathematics curriculum into arbitrary and necessary is based upon the philosophical roots of the notions of 'contingent' and 'necessary'. Kripke (1996) wrote that:

> If [something] is true, might it have been otherwise? [...] If the answer is 'no', then this fact about the world is a necessary one. If the answer is 'yes', then this fact about the world is a contingent one. (p. 36)

It is true that the *x* co-ordinate is written before the *y* co-ordinate. However, it might have been otherwise, a different decision could have been made with the *y* co-ordinate coming first. This is possible and the mathematics which would be based on such a convention would be just as consistent. So the fact that the *x* co-ordinate does come first is a contingent truth.

Nozick (1984) stated that:

> Let us state the principle of sufficient reason as: every truth has an explanation. For every truth *p* there is some truth *q* which stands in the explanatory relation E to *p*. [...] When any other truth holds without an explanation it is an arbitrary brute fact. (pp. 140-141)

There is no explanation why the *x* co-ordinate *must* come first, so this is an arbitrary fact, and indeed so is much of the rest of the mathematics curriculum on co-ordinates. Many chapters in textbooks are concerned with students knowing how to draw and label axes, how to write down co-ordinates, knowing that the *x* co-ordinate comes before the *y*, knowing the co-ordinate of a given point and knowing how to mark a point given its co-ordinates. These are all arbitrary and I suggest that mathematics does not lie with the arbitrary, but is found in what is necessary. *Before reading on, can you think of something within the curriculum on co-ordinates which is necessary and not arbitrary?*

I am concerned that little time is spent on what is necessary and so much time is spent on memorising and practising conventions. Mathematics is concerned with properties - and properties can be worked out or found out. This implies that much of the chapters on co-ordinates are not concerned with mathematics.

What mathematics does lie within the heading of co-ordinates? There is the awareness that a position cannot be described without starting from somewhere: an origin is required. This is not something a teacher needs to inform a student about; students can become aware of this through a suitably constructed task. Some form of base vectors (not necessarily at right angles) or their equivalents (such as angles in the case of polar co-ordinates) are also necessary, although, of course, they need not be known by that name.

These are some aspects of where mathematics lies within the topic of co-ordinates, rather than with the practising of conventions. I am not saying that the acceptance and adoption of conventions is not important within mathematics classrooms, but that it needs to be realised that this is not where mathematics lies. So I am left wondering about the amount of classroom time given over to the arbitrary compared with where the mathematics actually lies.

Approaches to teaching and their consequences

How might a teacher work on a given topic with a class given this division of the curriculum into arbitrary and necessary? For example, when carrying out a task involving throwing two six-sided dice a number of times and finding which total occurred most, one student, Sam, said to the teacher that he did not know the mathematical name given to the score which occurs most often. The teacher replied that Sam was there in a previous lesson when this was mentioned and that he should think about it.

I wondered what there was to 'think about'. The name - 'mode' - will either be remembered or not, and Sam is indicating that he does not remember. There is nothing here which can be worked out. The only options for him are to remember (which he didn't), or to be informed by someone else. Inviting students to 'think about it' is appropriate for what is necessary, but not for what is arbitrary. Since the name was not memorised on this occasion, an issue for the teacher was how students could be helped to memorise. For the arbitrary, a teacher's role is to assist memory. For what is necessary, a teacher does not need to inform students, since what is necessary can be worked out: a teacher's role is to work within the realm of awareness rather than memory. For example, the 'fact' that the internal angles of a triangle add up to half a full turn is something which students can become aware of themselves. The role for the teacher is to provide a task to help students to educate their own awareness of the angles inside a triangle. So, the teacher's role is to educate their students' awareness, rather than give them something to memorise (see Figure 6).

	Student	Teacher	Mode of teaching
Arbitrary	*All* students *need* to be informed of the arbitrary by someone else	A teacher *needs* to inform students of the arbitrary	*Assisting memory*
Necessary	*Some* students *can* become aware of what is necessary without being informed of it by someone else	A teacher does not need to inform students of what is necessary	*Educating awareness*

Figure 6 Modes of teaching

If a teacher decides to inform students of some mathematics content which is necessary, then they are treating it as if it is arbitrary, as if it is something which needs to be told. For example, if a teacher stated that the angles inside a triangle add up to half a full turn rather than offering a task for students to become aware of this, then students are left with having to accept what the teacher says as true. In this case, it becomes just another 'fact' to be memorised. I call this *received wisdom.*

It is possible for students to use their awareness to try to work out for themselves why this received wisdom is true. If a student succeeds, then this becomes a necessary fact and rightfully returns to the realm of awareness. All too often, however, a student just accepts this received wisdom and treats it as something to be memorised or, indeed, forgotten.

In a lesson I observed, some 14-15-year-olds were working on solving simultaneous equations, and one student was having difficulties with re-arranging an equation. He had written:

$$x - y = 2$$

$$y = 2 - x$$

I asked him about the '-' sign in front of the y and his response was to re-write the second equation as:

$$y = 2 + x$$

I said that I felt he had done the correct thing when taking away the x, but that there was still a '-' sign in front of the y. I wrote a '-' in front of the y in the original second equation:

$$-y = 2 - x$$

He then changed both the subtractions to additions saying "two negatives make a positive":

$$+y = 2 + x$$

This is one example of a student remembering some received wisdom - "two negatives make a positive" - but not remembering the situations in which this received wisdom is appropriate. This is a phrase he has remembered, but he has not got the awareness to accompany the memorised phrase. Rather than basing his actions on a mathematical awareness of inverse, his actions are informed by a memory of something to 'do' when there are two negatives present.

Transformations of equations are concerned with what is necessary and a teacher providing such a phrase turns such awareness into received wisdom which a student may then try to memorise. The problem with memory is that it gives the opportunity to forget. In this case, the phrase is remembered, but the associated situation it relates to (which is relatively complex) is forgotten.

Received wisdom may be accompanied by an explanation of why something is true. A teacher may explain why the angles in any triangle add up to half a full turn. The fact that a teacher gives an explanation does not mean that students have the awareness necessary to understand this explanation, or will do the required work to be in a position where they too know why this *must* be the case. A careful explanation may increase the possibility that some students will be able to use the awareness they have in order to come to realise why this 'fact' is true. Other students may not have sufficient awareness, or not choose to use the awareness they have to get themselves into such a position. For these students, the fact that the interior angles of any triangle add up to half a full turn remains received wisdom and in the realm of memory, despite the teacher's efforts.

Poincaré (undated) considered the following scenario:

> In the same way our pupils imagine that they know it when they begin to study mathematics seriously. If, without any other preparation, I come and say to them: "No, you do not know it; you do not understand what you imagine you understand; I must demonstrate to you what appears to you evident;" and if, in the demonstration, I rely on premises that seem to them less evident than the conclusion, what will the wretched pupils think? They will think that the science of mathematics is nothing but an arbitrary aggregation of useless subtleties; or they will lose their taste for it; or else they will look upon it as an amusing game. (p. 128)

A teacher's explanation is often based upon the teacher's awareness, and so may use things which students do not find *evident* - things which are not in the students' awareness - and so the explanation will not be one which will help those students to educate their own awareness. As Poincaré points out, for many students mathematics can become "an arbitrary aggregation of useless subtleties" or just a game with symbols (although I doubt it is often considered 'amusing').

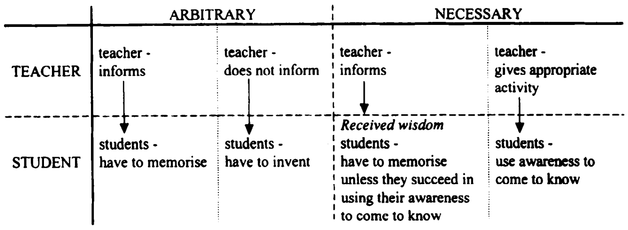


Figure 7 A summary of teacher choices and consequent student way of working

Figure 7 gives a summary of the choices available to a teacher with the arbitrary and the necessary, and the consequent result in terms of the way in which students have to work. I will review parts of this summary by considering Merttens' (1995) list of teaching strategies, which includes *Giving instruction*:

> **Giving instruction:** This is the strategy most aligned with traditional teaching, and perhaps the one which has come in for most adverse criticism over the last twenty years. [...] Instruction, or the provision of procedures, is the practice which occurs when someone:
>
> 1. gives a recipe;
> 2. gives a set of directions;
> 3. shows the way to move along a numbered track;
> 4. explains how to do a multiplication sum;
> 5. explains the rules of *Monopoly;*
> 6. demonstrates how to write a series of joined-up letters;
> 7. shows how to put on a life jacket;
> 8. gives the procedure for crossing the road safely.
>
> (p. 7, *my numbering*)

Looking at this list and analysing it in terms of arbitrary and necessary, I claim that all except number 4 are arbitrary (I am not considering number 3, as I do not feel sufficiently clear about what Merttens meant by this example to comment upon it). This means that it is wholly appropriate for instruction to be a strategy used for these examples, since students will not know a *particular* recipe, how to play *Monopoly* (in a *particular* way), or write joined-up letters (in a particular way), etc., unless they are informed. As indicated in Figure 7, if a teacher chooses not to inform students of these things, then students are still perfectly capable of inventing recipes, ways to play *Monopoly*, or how to write joined-up letters, etc.

Example 4, however, is different since multiplication is necessary. Two times three is six and not five. (I am not referring to the words - the signifiers - *two, three, multiplied*, etc., but to the signified - *twoness*, for want of a better expression, the usual operation associated with the word *multiplied*, etc.) Students can invent what two times three is - for example seven - but they may be wrong! Although there are alternatives to how a multiplication sum is carried out, the issue of multiplication of numbers is necessary and so how a multiplication is carried out will either be mathematically correct or not. If students are informed of how to do a multiplication sum, then this is received wisdom.

Merttens goes on to write:

> To instruct children is to give them a series of "now do this, then do that" procedures. ***It is also to credit them with their own intelligence.*** It is to assume that, with our help, they will utilize these procedures as and when appropriate, that they will have the intelligence not only to adopt but to adapt them, phrasing them in their own terms and for their own reasons, articulating them (in all senses of the word) in their own contexts; [...] The much-talked-of *understanding* will either come as they use the procedure, or later on, or even not at all because they never have the need to relate that particular algorithm to any other aspects of the subject. To understand, in this sense, is to translate, to incorporate what one has been given into one's own story. (p. 7, emphasis in original)

Students may well go on to use "their own intelligence" by becoming aware of why certain procedures must give correct answers, in which case, the received wisdom will become something which is necessary and so "incorporate what one has been given into one's own story". However, I have seen too many classrooms where students are not given the time or indeed the encouragement to work on trying to *understand* the received wisdom they have been offered. Too often, mathematics lessons appear to be about receiving the teacher's wisdom and practising how to replicate it, before moving on to the next item of received wisdom the teacher passes on. So, I feel there are other reasons for why students do not *understand* procedures about necessary aspects of the mathematics curriculum other than never needing "to relate that particular algorithm to any other aspects of the subject".

Givens - assumed properties

When using awareness to find out that something is necessary, there may be certain information provided other than arbitrary names and conventions. For example, in Figure 8 the task might be to find the area and perimeter of the rectangle. The area and perimeter are necessary, since they can be worked out, but only because there have been properties already given, such as the length of the sides of the rectangle. These properties I describe as *givens*.

Thus, what is necessary comes as a consequence of certain accepted givens. If insufficient properties were given to determine the area, then they could be created - with either a number chosen or a label assigned. For example, the following statement about area:

$$\text{area} = L \times B$$

can only be articulated with the assigning of the unknowns L and B. It should be noted here that there is a combination of arbitrary labels (here, L and B) and the properties which they are labelling - the actual lengths of the sides of the rectangle. It is the former which are arbitrary and the latter which are the givens. The arbitrary (the labels) is adopted and the givens (the properties) accepted and worked with in order to find what is necessary.

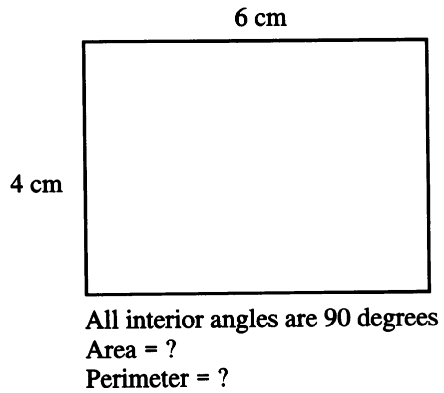


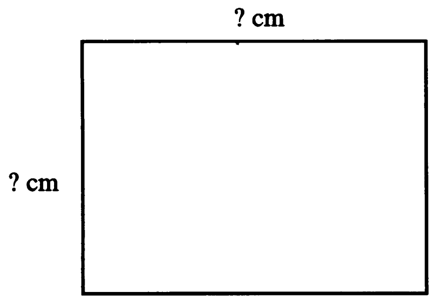
Figure 8 A traditional question

Givens can become known to a student in three ways:

- firstly, a student can 'receive' them, such as verbally from a teacher or through written text (as with Figure 8), etc.;
- secondly, a student can observe them through their senses, such as seeing that one side of a rectangle is longer than another, or feeling that the corner of a room is more than 90 degrees, etc.;
- thirdly, students can create their own givens, such as allocating the property of the length of a side of a rectangle the label L, or inventing an equation to try to solve, etc.

Givens are required in order that other things become necessary. However, the givens in Figure 8, such as the length of the sides of the rectangle, are themselves properties which might have been necessary had the area and perimeter been given in the first place (see Figure 9).

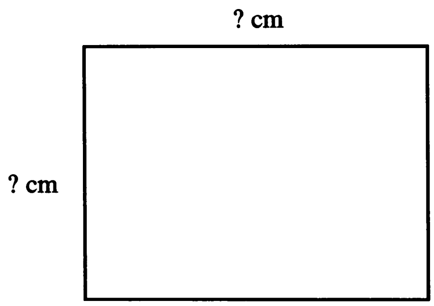
If some properties are given but not sufficient to determine the lengths, such as in Figure 10, then this does not stop certain things being worked out from the facts which are given. For example, I can say that the smallest the perimeter can be is $4 \times \sqrt{24}$. Perimeter remains a part of the



All interior angles are 90 degrees
Area = 24 cm
Perimeter = 20 cm

Figure 9 An alternative question

mathematics curriculum which is about properties and I can use my awareness to derive new certainties within this area of the curriculum based on related givens.



All interior angles are 90 degrees
Area = 24 cm
Perimeter = ? cm

Figure 10 What can be known for sure now?

Although givens are properties, they lie in the realm of memory, since these are assumed facts rather than derived certainties. As such, they cannot be worked out and so need to be memorised if they are to be available in the future (without needing to be informed of them again). So, although perimeter is an aspect of the curriculum which is necessary, a given particular perimeter such as in Figure 9 is one which I will have to memorise. So, the properties given within particular mathematical questions have to be memorised, whilst other related properties can be derived through awareness from these memorised givens.

Generates – generating new possibilities

Although the arbitrary requires memory in order to be retained, awareness can still be used with the arbitrary. This can be done in two ways. Firstly, an awareness concerning properties can be based upon adoption of a convention. For example, having adopted the convention that there are 360 degrees in a whole turn and that measurement of turn is based on a linear scale (both arbitrary), then I can use the awareness I have about linearity to say *if I halve this, then I halve that.* This leads me to be able to say definitely that there are 180 degrees in a half a full turn – a certainty worked out through awareness from the original adopted convention.

My use of the word 'certainty' here is based upon the premise *if I adopt this convention, then* there is a property I can state about a half-turn which is true (and I did not need to be informed about it). Of course, if a different convention is adopted, then the property may no longer be true. Ayer (1962) gives an example from within philosophy:

> For apart from the fact that they [*a priori* propositions] can properly be said to be true, which linguistic rules cannot, they are distinguished also by being necessary, whereas linguistic rules are arbitrary. At the same time, if they are necessary it is only because the relevant linguistic rules are presupposed. Thus, it is a contingent, empirical fact that the word "earlier" is used in English to mean earlier, and it is an arbitrary, though convenient, rule of language that words that stand for temporal relations are to be used transitively; but, given this rule, the proposition that, if A is earlier than B and B is earlier than C, A is earlier than C becomes a necessary truth. (p. 17)

The *if ..., then this must be so* scenario is at the basis of working mathematically to establish new certainties – the necessary.

The second way of using awareness with the arbitrary is based upon being creative with the conventions themselves: *if ..., then this could be so*. For an example, consider number names. The names used for our numbers are arbitrary, as are the conventions of how these names are used. *One, two, three, hundred, thousand*, etc., are arbitrary as they could equally well be *un, deux, trois, cent, mille*, etc. The convention in English is that 21 is said with the highest value digit first – *twenty-one* – while in German it is said with the lowest value digit first – *ein-und-zwanzig*. So the way in which the words are combined is also arbitrary. Yet, having adopted the names and conventions in English, I can use my awareness to generate new number names. For example, say the number below out loud:

4280381

I claim that you have never said this number before in your life nor heard it said. As a consequence, you cannot have memorised how to say it. The ability to generate new number names from adopted conventions is in the realm of awareness. Yet these are not necessary, since they are still only names. So I describe such things as *generates*. These have been generated from names and conventions, using awareness. Issues of 'right' or 'wrong' are not appropriate, since alternatives are always possible; it is only a matter of whether alternatives are accepted within a certain culture. What was accepted in the past, such as the word 'billion' meaning 1,000,000,000,000 in the UK, can become changed over time, as indeed 'billion' is used now to mean 1,000,000,000 in the UK.

As well as awareness being used to work with conventions to produce names for numbers never said before, some conventions can be explored to some extremities which are not usually carried out within a culture. For example, a child saying the following number names: *one hundred, two hundred, three hundred, ...* may continue and say: *eight hundred, nine hundred, ten hundred, eleven hundred, ...*

There is nothing 'wrong' with this and indeed I have heard such usage on the television and radio (as well as it being common in terms of naming years, e.g. nineteen hundred and ninety-nine). However, this can be explored further: *two hundred and forty seven hundred* for 24,700; or *twenty three point four tenths* for 2.34. These are not heard on the radio, and yet they only explore an accepted convention further than is usual. Pedagogically, this exploration has its uses, as it can be helpful to be flexible about ways of viewing and naming numbers.

Conventions can also be extended: for example, the notation for two-dimensional co-ordinates can be extended to three, four and more dimensions, and so offer a way to work with otherwise conceptually complex scenarios. Conventions can also be combined, such as writing $\frac{1}{2}(3, 4)$ to represent $(1\frac{1}{2}, 2)$. These are ways in which someone can use their awareness to generate new possibilities within the world of conventions. Not all of them will be accepted within the mathematics community, but they are examples of using awareness within this area. The ability to generate such possibilities reduces the otherwise overwhelming demand on memory. Borges (1985) created a character, Funes, who never forgot anything and used to create a new name for every number. Thankfully for us, the demands on memory are less taxing through the use of our ability to generate names for numbers from relatively few words.

Summary

An overview of the dynamics discussed in this article is contained in Figure 11. Viewing the mathematics curriculum in terms of those things which can be worked out by someone (necessary) and those things which everyone needs to be informed about (arbitrary) can clarify the roles both teacher and students have within the complexities of teaching and learning. The arbitrary is concerned with names and conventions and students have no choice but to memorise the arbitrary, and a teacher will have to inform them of what is arbitrary. I have indicated this in Figure 11 as the student having 'received' the arbitrary. The only other option is for the teacher to refuse to inform students and leave them to invent something instead. This is perfectly possible and can be desirable at times: however, it does not change the fact that students will still need to be informed at some time in the future if they are to be included within a mathematics community which communicates through adopted conventions.

As Mandler (1989) points out:

> The good theory can well say with Humpty Dumpty: "When I use a word, it means just what I choose it to mean - neither more nor less." However, the history of the social sciences is strewn with abandoned concepts and terms that have failed to heed a corollary that Humpty Dumpty never told us about: Once you choose a word to mean something (exactly), then you have to start convincing other people to use it that same way; otherwise, monologues will never be replaced by dialogue and consensus. (p. 237)

Students are unlikely to convince the mathematics community to change the names and conventions already established, even if they had the platform to attempt to do this. So it is the student who, perhaps unfairly, needs to accept and adopt in order to communicate with the mathematics community.

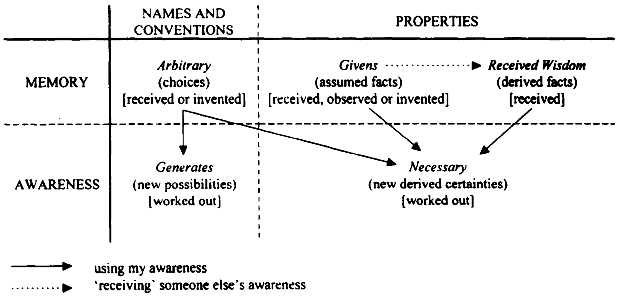


Figure 11 An overview

The necessary is about properties, and one possibility is for students to 'receive' properties through a teacher informing them just as for the arbitrary. However, this turns the necessary into received wisdom and students may well treat this as something else to be memorised. Indeed, they will have no other choice unless they are able and willing to do the work necessary to become aware of the necessity of this received wisdom. Some students may be able to do this work, in which case the received wisdom will become a derived certainty and be known through awareness rather than memory.

Another choice for a teacher is to provide a task which will make properties accessible through awareness. An appropriate task will help these properties to be more accessibly known through awareness than if the teacher informed students of them and left the students to their own devices to work out why they must be so.

A teacher taking a stance of deliberately not informing students of anything which is necessary is aware that developing as a mathematician is about educating awareness rather than collecting and retaining memories. Furthermore, this stance clarifies for the students the way of working which is appropriate for any particular aspect of the curriculum - the arbitrary has to be memorised, but what is necessary is about educating their awareness.

> *If I'm having to remember ..., then I'm not working on mathematics.*

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